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THE STABILITY OF THE EQUILIBRIUM ELLIPSOIDS OF A ROTATING LIQUID†

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The stability of the Riemann equilibrium ellipsoids, in the class of perturbations which satisfy the Dirichlet assumptions, is investigated using Rumyantsev's method [1]. It is shown that the equations of motion of the Dirichlet liquid ellipsoid are Hamiltonian on each combined level of the moment and circulation integrals, which corresponds to well-known results [2], although it is not a consequence of them. This fact provides, generally speaking, additional possibilities for solving the problem of determining the instability region. In the parameter space of each of the two families [3, 4] of Riemann ellipsoids, the region U for which almost all the equilibrium ellipsoids belonging to it are unstable is determined in explicit analytical form. It is shown that the stability region can be specified in explicit analytical form in both cases.

The stability of the Maclaurin [1] and Jacobi [5] ellipsoids were investigated in a similar way earlier.

1. FORMULATION OF THE PROBLEM

Consider an ideal homogeneous incompressible liquid, the particles of which are attracted to one another in accordance with Newton's. In the functional space of the system, which describes its dynamics, provided that the initial perturbation satisfies the Dirichlet assumptions that the surface of the liquid is ellipsoidal and that its velocity field is uniformly vortical, we will change to a system of ordinary differential equations for the vorticity components $(2\omega_1(t), 2\omega_2(t), 2\omega_3(t))$, the semi-axes of the ellipsoid (a, b, c) and the components of the angular velocity (p, q, r) in a moving frame of reference [1].

We will present this system of ordinary differential equations here in the form that will be most convenient later, obtained from the initial system of equations [1] taking into account the condition for the liquid volume to be constant, which is a consequence of the incompressibility equation: abc = const, where, without loss of generality, we can assume const = 1. We have

$$\frac{d}{dt}(A_1p+A_2\omega_1)+q(C_1r+C_2\omega_3)-r(B_1q+B_2\omega_2)=0 \tag{1.1}$$

$$(pqr,\ \omega_1\omega_2\omega_3,\ ABC)$$

$$\frac{d}{dt}\left(\frac{\omega_1}{a}\right)-\frac{2a}{a^2+b^2}r\omega_2+\frac{2a}{a^2+c^2}q\omega_3+\frac{2a(c^2-b^2)}{(a^2+c^2)(a^2+b^2)}\omega_2\omega_3=0 \tag{1.2}$$

$$(123,\ abc,\ pqr)$$

$$\ddot{a}\left(a+\frac{1}{b^2a^3}\right)+\frac{\ddot{b}}{a^2b^3}-\frac{2\dot{a}\dot{b}}{a^3b^3}-\frac{2\dot{a}^2}{a^4b^2}-\frac{2\dot{b}^2}{b^4a^2}=(R-\tilde{\omega}_z)\frac{1}{a^2b^2}-(P-\tilde{\omega}_z)a^2 \tag{1.3}$$

$$\ddot{b}\left(b+\frac{1}{a^2b^3}\right)+\frac{\ddot{a}}{b^2a^3}-\frac{2\dot{a}\dot{b}}{a^3b^3}-\frac{2\dot{a}^2}{a^4b^2}-\frac{2\dot{b}^2}{b^4a^2}=(R-\tilde{\omega}_z)\frac{1}{a^2b^2}-(Q-\tilde{\omega}_y)b^2 \tag{1.4}$$

where

$$\tilde{\omega}_x=\frac{\ddot{a}}{a}-\omega_x=\frac{a^2-c^2}{(a^2+c^2)^2}(a^2+3c^2)q^2+\frac{a^2-b^2}{(a^2+b^2)^2}(a^2+3b^2)r^2-\frac{4(a^2-c^2)c^2}{(a^2+c^2)^2}q\omega_2-$$
$$-\frac{4(a^2-b^2)b^2}{(a^2+b^2)^2}r\omega_3+\frac{4a^2b^2}{(a^2+b^2)^2}\omega_3^2+\frac{4a^2c^2}{(a^2+c^2)^2}\omega_2^2$$

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$$(xyz, \quad abc, \quad pqr) \tag{1.5}$$

$$A_1 = \frac{M}{5}\frac{(b^2-c^2)^2}{b^2+c^2}, \quad A_2 = \frac{4M}{5}\frac{b^2c^2}{b^2+c^2}(ABC, abc)$$

$$P = \frac{2}{a}\left[\frac{\partial}{\partial a}\hat{H}(a,b,c)\right](PQR, \; abc) \tag{1.6}$$

$$\hat{H} = \frac{3M}{4}\int\frac{d\lambda}{\sqrt{\varphi(\lambda)}}, \quad \varphi(\lambda) = (a^2+\lambda)(b^2+\lambda)(c^2+\lambda)$$

and M is the mass of liquid.

Here and everywhere henceforth in all the formulae, apart from formulae (1.6), (2.5) and (2.10), we will assume $c = 1/(ab)$; the integration is carried out from zero to infinity.

If we take into account the Dirichlet conditions and the known form of the solution of Laplace's equation, Eqs (1.1) of system (1.1)–(1.4) follow from the theorem on angular momentum, Eqs (1.2) follow from the Helmholtz equations for a vortex in moving axes, and Eqs (1.3) and (1.4) follow from Euler's hydrodynamic equations in moving axes [1].

System (1.1)–(1.4) has three integrals: energy, angular momentum and constancy of the vortex intensity [1]

$$\frac{1}{2}(A_1p^2+B_1q^2+C_1r^2+A_2\omega_1^2+B_2\omega_2^2+C_2\omega_3^2)+W+\frac{M}{10}(\dot{a}^2+\dot{b}^2+\dot{c}^2) = \text{const} \tag{1.7}$$

$$(A_1p+A_2\omega_1)^2+(B_1q+B_2\omega_2)^2+(C_1r+C_2\omega_3)^2 = \text{const} \tag{1.8}$$

$$(\omega_1/a)^2+(\omega_2/b)^2+(\omega_3/c)^2 = \text{const} \tag{1.9}$$

where

$$W = -\frac{2\cdot 3}{5\cdot 4}M^2\int\frac{d\lambda}{\sqrt{\varphi(\lambda)}} \tag{1.10}$$

Riemann showed [3, 4], that the ellipsoidal equilibrium configurations of a rotating liquid (for motions with uniform deformation) can exist, but only in the case when the corresponding velocity field of the liquid particles is the superposition of rigid-body rotation and internal uniformly vortex motions. Hence, each such equilibrium ellipsoid automatically satisfies the Dirichlet assumptions, and a certain equilibrium position of system (1.1)–(1.4) corresponds to it. On the other hand, any perturbation, which satisfies the Dirichlet assumptions at the initial instant, will also satisfy them any subsequent instant of time (its dynamics in this case is determined by system (1.1)–(1.4)).

The question therefore naturally arises here of the stability of the Riemann equilibrium ellipsoids in the class of perturbations which satisfy the Dirichlet assumptions, which is equivalent to the Lyapunov stability of the corresponding equilibria of system (1.1)–(1.4).

Riemann proved [3] that the set of all equilibria of system (1.1)–(1.4) is a union of two subsets ("families"): for the ellipsoids of the first of these the angular velocity and vorticity vectors of the internal motions lie in one of the principal planes of the ellipsoid, while for the ellipsoids of the second family these vectors are collinear and are directed along one of the axes [3, 4].

The analytical solution of the problem of the stability of the Riemann equilibrium ellipsoids in the class of perturbations, which satisfy the Dirichlet assumptions, were only obtained for certain special cases of the ellipsoids of the second family. Riemann's and Lyapunov's conclusions are the main results in this area of knowledge.

Riemann investigated the stability of ellipsoids lying on a certain curve in a two-dimensional manifold of ellipsoids of the second family, but in a narrow class of perturbations which, in addition to the Dirichlet assumptions, also satisfy the requirements of conservation: (1) the orientations of the angular velocity and vorticity vectors and (2) the values of integrals (1.8) and (1.9).

Lyapunov gave a solution of the stability problem for the cases of the Maclaurin and Jacobi ellipsoids, which are special cases of the Riemann ellipsoids of the second family [4]; apparently none of the remaining Riemann ellipsoids was considered by Lyapunov.

Lyapunov's proofs were based on proofs that a strict minimum was reached on the functional of the changed potential energy in the steady rotation considered. Note, however, that the stability problem, in the class of perturbations which satisfy Dirichlet's assumptions, can also be solved differently using Lyapunov's second method, since system (1.1)–(1.4) is a system of ordinary differential equations (with a finite number of degrees of freedom).

A corresponding method was presented in [1]; it can obviously also be used to investigate the stability of any Riemann ellipsoids.

In this paper we describe some results obtained when solving the problem of the stability of Riemann ellipsoids using a method which is a development of Rumyantsev's method [1].

Henceforth we mean by the stability of the equilibrium ellipsoids, their stability in the class of perturbations which satisfy Dirichlet's assumptions, i.e. their Lyapunov stability as equilibria of system (1.1)–(1.4).

In Section 3 we give a general scheme of the method used to solve the problem. The stability region S is determined using Lyapunov's second method; here it turns out to be possible to specify it in an explicit analytical form both in the parameter space of the first family of Riemann ellipsoids (Section 4), and in the parameter space of the second family (Section 5).

The fact that system (1.1)–(1.4) is Hamiltonian on any combined level of the momentum integral (1.8) and the circulation integral (1.9), the proof of which is given in Section 2, enables us, in this case, to use a certain general scheme of analytical determination in the parameter space of the equilibria of the family of Hamiltonian systems of the region U, such that almost all the equilibria belonging to it are unstable (Section 3). The corresponding regions U, both for the first (Section 4) family of Riemann ellipsoids and for the second (Section 5), can be specified in explicit analytical form by an appropriate choice of the coordinates in the parameters spaces of these families.

2. THE STRUCTURE OF SYSTEM OF EQUATIONS (1.1)–(1.4)

It was shown in [2] that the equations of motion of any finite-dimensional mechanical system with holonomic constraints and potential forces can be represented in Hamiltonian form. As t turns out, this is also true for system (1.1)–(1.4); this fact corresponds to the well-known results in [2], although it is not a direct consequence of them: a priori it is not clear whether system (1.1)–(1.4) can be identical with a system with holonomic constraints and potential forces.

We will show that system (1.1)–(1.4), after the diffeomorphism

$$(a,b,\dot a,\dot b,p,q,r,\omega_1,\omega_2,\omega_3)\to\{z\}=\{a,b,p_a,p_b;G_1,G_2,G_3,l_1,l_2,l_3\} \tag{2.1}$$

given by the relations

$$\begin{aligned} G_1&=\frac{M}{5}\frac{(b^2-c^2)^2}{b^2+c^2}p+\frac{4M}{5}\frac{b^2c^2}{b^2+c^2}\omega_1 \quad (123,\ abc,\ pqr)\\ l_1&=-\frac{2M}{5}bc\omega_1 \quad (123,\ abc)\\ p_a&=\frac{M}{5}\left(\dot a\left(1+\frac{1}{a^4b^2}\right)+\frac{\dot b}{a^3b^3}\right);\quad p_b=\frac{M}{5}\left(\dot b\left(1+\frac{1}{b^4a^2}\right)+\frac{\dot a}{a^3b^3}\right)\end{aligned} \tag{2.2}$$

changes to the system

$$\dot z=\{z,H\} \tag{2.3}$$

Here $H(z)$ is the energy (1.7) in $\{z\}$ coordinates (2.1)

$$H=\mathcal{K}+\frac{5}{2M}\frac{[a^4b^4(p_a^2+p_b^2)+(ap_a-bp_b)^2]}{a^4b^4+a^2+b^2} \tag{2.4}$$

where

$$\mathcal{K}((a,b,c;)G_i,l_i)=\frac{5}{2M}\left[\frac{b^2+c^2}{(b^2-c^2)^2}G_1^2+\frac{a^2+c^2}{(a^2-c^2)^2}G_2^2+\frac{a^2+b^2}{(a^2-b^2)^2}G_3^2+\frac{4bc}{(b^2-c^2)^2}G_1l_1+\right.$$

$$+\frac{4ac}{(a^2-c^2)^2}G_2l_2+\frac{4ab}{(a^2-b^2)^2}G_3l_3+\frac{b^2+c^2}{(b^2-c^2)^2}l_1^2+\frac{a^2+c^2}{(a^2-c^2)^2}l_2^2+\frac{a^2+b^2}{(a^2-b^2)^2}l_3^2\Bigg]+W \tag{2.5}$$

The variable c in $\mathcal{H}$ (2.5) is assumed (for convenience) to be independent; {,} is the Poisson bracket on $C^\infty(Z)$, specified as follows:

$$\{G_i,G_j\}=\varepsilon_{kji}G_k,\quad \{l_i,l_j\}=\varepsilon_{kji}l_k,\quad \{a,p_a\}=\{b,p_b\}=1 \tag{2.6}$$

while the brackets between all the remaining pairs of phase variables are zero.

Starting from the form of the function H (2.4) and taking into account definition (2.6) we obtain that Eq. (2.3), corresponding to the variable $z = G_1$, has the form

$$\dot{G}_1=\frac{\partial H}{\partial G_2}(-G_3)+\frac{\partial H}{\partial G_3}G_2=\frac{5}{M}\left[\frac{a^2+b^2}{(a^2-b^2)^2}G_3+\frac{2ab}{(a^2-b^2)^2}l_3\right]G_2-$$
$$-\frac{5}{M}\left[\frac{a^2+c^2}{(a^2-c^2)^2}G_2+\frac{2ac}{(a^2-c^2)^2}l_2\right]G_3$$

and, by relations (2.2), converts into the initial variables in the first momentum equation (1.1).

Equation (2.3) for the variable $z = l_1$.

$$\dot{l}_1=\frac{5}{M}\left[\frac{a^2+b^2}{(a^2-b^2)^2}l_3+\frac{2ab}{(a^2-b^2)^2}G_3\right]l_2-\frac{5}{M}\left[\frac{a^2+c^2}{(a^2-c^2)^2}l_2+\frac{2ac}{(a^2-c^2)^2}G_2\right]l_3$$

in the initial variables acquires a form identical to the first Helmholtz equation (1.2).

Equations (2.3) for the semiaxes a and b convert, in the variables $(a, b, \dot{a}, \dot{b}, p, q, r, \omega_1, \omega_2, \omega_3)$, into the identities $da/dt = \dot{a}$ and $db/dt = \dot{b}$, as it should.

We will now show that the equation

$$dp_a/dt=-H_a$$

of system (2.3) is Eq. (1.3) in the initial variables.

Differentiating function (2.4) we obtain this equation in expanded form

$$\dot{p}_a=\left(-\frac{\partial\mathcal{H}}{\partial a}+\frac{c}{a}\frac{\partial\mathcal{H}}{\partial c}\right)\Bigg|_{c=1/(ab)}+\frac{5}{2M}\frac{\partial}{\partial a}\left(\frac{(ap_b+bp_a)^2}{a^4b^4+a^2+b^2}\right) \tag{2.7}$$

In the initial phase variables we have

$$\dot{p}_a=\frac{d}{dt}\left(\frac{M}{5}\dot{a}\left(1+\frac{1}{a^4b^2}\right)+\frac{M}{5}\frac{\dot{b}}{a^3b^3}\right)=-\frac{M}{5}\left(\frac{2\dot{a}^2}{a^5b^2}+\frac{\dot{b}^2}{a^3b^4}+\frac{3\dot{a}\dot{b}}{a^4b^3}\right)+$$
$$+\frac{M}{5}\left(\ddot{a}\left(1+\frac{1}{a^4b^2}\right)+\frac{\ddot{b}}{a^3b^3}-\frac{2\dot{a}^2}{a^5b^2}-\frac{2\dot{b}^2}{a^3b^4}-\frac{2\dot{a}\dot{b}}{a^4b^3}\right) \tag{2.8}$$

$$\frac{\partial}{\partial a}\left(\frac{(ap_b+bp_a)^2}{a^4b^4+a^2+b^2}\right)=-\frac{2M^2}{25}\left(\frac{2\dot{a}^2}{a^5b^2}+\frac{\dot{b}^2}{a^3b^4}+\frac{3\dot{a}\dot{b}}{a^4b^3}\right) \tag{2.9}$$

Finally, differentiating the function $\mathcal{H}$ (2.5) with respect to a and c and taking relations (1.6) into account, we obtain, after reduction

$$\frac{\partial\mathcal{H}}{\partial a}=\frac{M}{5}a(P-\tilde{\omega}_x)\quad \frac{\partial\mathcal{H}}{\partial c}=\frac{M}{5}c(R-\tilde{\omega}_z) \tag{2.10}$$

where $\tilde{\omega}_x$ and $\tilde{\omega}_z$ are the functions (1.5).

Now substituting expressions (2.8)–(2.10) into (2.7), we obtain Eq. (1.3).

In exactly the same way we obtain that Eqs (2.3) for the variables $z = G_2, G_3$ convert, in the initial variables, into the corresponding equations (1.1); Eq.(2.3) for the variable $z = l_2, l_3$ converts into the last two of the Helmholtz equation (1.2), while Eq.(2.3) for p_b converts into Eq.(1.4).

Hence, system (1.1)–(1.4), which describes the dynamics of the Dirichlet liquid ellipsoid turns out to be coincident with the Hamiltonian system of the form (2.3) (apart from the diffeomorphism (2.1)).

Here at any combined level

$$M = \left\{ z : \sum_i G_i^2 = \sum G_{0i}^2,\ \sum l_i^2 = \sum l_{oi}^2,\ (i = 1,2,3) \right\}$$

of integrals (1.8) and (1.9), system (1.1)–(1.4) is a Hamiltonian system, since the restriction {,}' of the bracket {,} (2.6) to $C^\infty(M)$ is non-degenerate. This follows directly from the from of (2.6), taking into account the fact that the annulet of the Poisson bracket, corresponding to the usual commutator in the so (3) $\{m_1, m_2, m_3\}$ algebra, is generated exactly by the function $m_1^2, + m_2^2, + m_3^3$.

3. A GENERAL SCHEME FOR DETERMINING THE STABILITY AND INSTABILITY REGIONS

Suppose now that z_0 is a certain equilibrium ellipsoid. Consider system (1.1)–(1.4) at the level

$$M_{z_0} = \left\{ z : \sum G_i^2 = G_i^2(z_0),\ \sum l_i^2 = l_i^2(z_0) \right\}$$

$$M_{z_0} = M_{z_0}\{\tilde{z}\},\quad (\tilde{z}) = (a, b, p_a, p_b; G_1, l_1, G_2, l_2)$$

We have

$$\dot{\tilde{z}} = \{\tilde{z}, h(\tilde{z}; (z_0))\}' \tag{3.1}$$

Here $h(\tilde{z}; (z_0))$ is the restriction of the function (2.4) to the level M_{z_0}, and the notation (z_0) indicates the parametric dependence of the function $h(\tilde{z}; (z_0))$ on the coordinates of the point z_0 in the space of the set of equilibrium ellipsoids considered. The bracket {,}' is specified in $\tilde{z}$ coordinates on M_{z_0}, taking definitions (2.6) into account, by the relations

$$\begin{gathered} \{a, p_a\}' = \{b, p_b\}' = 1 \\ \{l_1, l_2\}' = -l_3(z_0)\left(1 + \frac{l_1^2(z_0) + l_2^2(z_0) - l_1^2 - l_2^2}{l_3^2(z_0)}\right)^{1/2} \\ \{G_1, G_2\}' = -G_3(z_0)\left(1 + \frac{G_1^2(z_0) + G_2^2(z_0) - G_1^2 - G_2^2}{G_3^2(z_0)}\right)^{1/2} \end{gathered} \tag{3.2}$$

while the brackets for all the remaining pairs of phase variables (z) are zero.

The condition

$$h^{(2)}(\delta\tilde{z}; (z_0)) = d^2 h(\tilde{z}; (z_0))\,|_{\tilde{z}=\tilde{z}_0} > 0 \tag{3.3}$$

equivalent to the condition that the Lyapunov function from integrals (1.7)–(1.9) of system (1.1)–(1.4) exists, is obviously the sufficient condition for the ellipsoid z_0 considered to be stable.

When condition (3.3) is satisfied, the corresponding Lyapunov function will be, for example, the function

$$V(z) = (H(z) - H(z_0))^2 + \left(\sum G_i^2 - G_i^2(z_0)\right)^2 + \left(\sum l_i^2 - l_i^2(z_0)\right)^2$$

The question naturally arises as to what extent (3.3) can also be the necessary stability condition.

Condition (3.3) implicitly specifies the stability region S in the parameter space of the ellipsoids considered.

The fact that system (3.1) is Hamiltonian also enables one, moreover, to specify the region U in it such that almost all the ellipsoids belonging to it are unstable.

Suppose $\{y(z_0)\}$ are canonical coordinates, which always exist in view of the Darboux theorem, in which system (3.1), linearized in the neighbourhood of the point z_0, has the standard Hamiltonian form

$$\dot{y} = \operatorname{sgrad}\,(K^{(2)}(y;\,(z_0))) \tag{3.4}$$

where

$$K^{(2)}(y;(z_0)) = h^{(2)}(\delta\bar{z};(z_0))\,|_{\{\delta\bar{z}\}\to\{y\}}$$

In the parameter space of the family of equilibrium ellipsoids considered, we define the region D as the region occupied by those ellipsoids z_0 for which the limitation of the corresponding form $K^{(2)}(y;\,(z_0))$ on some (any) Lagrange plane in a symplectic manifold $\{y(z_0)\}$ is positive-definite.

Then, for any ellipsoid $z_0 \in D$ of general position, condition (3.3) is the stability criterion.

We will show the necessity. Suppose the ellipsoid $z_0 \in D$ of general position is the stable equilibrium position of system (1.1)–(1.4). Then all the eigenvalues of the matrix of the form $K^{(2)}(y;\,(z_0))$ are pure imaginary and different. In this case, as is well-known, a symplectic transformation $\{y\} \to \{p_i, q_i\}$ exists by which the form $K^{(2)}(y;\,(z_0))$ is reduced to the canonical form

$$\sum_{i=1}^{4} \alpha_i(p_i^2 + q_i^2), \quad \alpha_i \neq 0 \tag{3.5}$$

where p_i and q_i are conjugate phase variables corresponding to one another. If in the form (3.5) at least one $\alpha > 0$, then in the symplectic manifold $\{p_i, q_i\}$ no Lagrange plane exists the restriction to which of function (3.5) would be positive-definite. Hence, starting from the above assumption, taking into account the definition of the region D and the fact that, with a symplectic transformation the Lagrange plane transforms into a Lagrange plane, we obtain that $\alpha_i > 0$ for all i. Consequently, the stability of the ellipsoids z_0 implies that the form $K^{(2)}(y;\,(z_0))$, and of course also $h^{(2)}(\delta\bar{z};\,(z_0))$, are positive-definite, which it was required to prove.

It is obvious that almost all ellipsoids $z_0 \in U$, $U = D \setminus \bar{S}$, where D and S are defined above, are unstable.

Note also that in order for the ellipsoid z_0 to belong to the region D defined above, it is not entirely necessary to satisfy, in addition to the condition

$$K^{(2)}(y;\,(z_0))|_L > 0$$

where L is a certain Lagrange plane, the condition that the form $K^{(2)}(y;\,(z_0))$ should be "split" into two, one of which is $K^{(2)}(y;\,(z_0))\,|_L$.

In order words, it is not at all necessary that system (3.1) should be identified with a certain "natural" system.

Note that the discussion presented above is fairly general and can be used to solve the problem of the analytical determination of the instability region in the space of equilibria of any family of Hamiltonian systems.

4. THE STABILITY OF EQUILIBRIUM ELLIPSOIDS OF THE FIRST FAMILY OF RIEMANN ELLIPSOIDS

Riemann showed [3] that equilibrium ellipsoids only exist in two cases:

1. when one of three pairs (ω_1, p), (ω_2, q) and (ω_3, r) (or (G_i, l_i), $i = 1, 2, 3$) has both zero components: without loss of generality – the first: $l_1 = G_1 = 0$,
2. when the zero components have two of these pairs: ($l_1 = G_1 = 0$ and $l_2 = G_2 = 0$).

Consider the first of these families of equilibrium ellipsoids $P^2_{(1)}$. It is formed by the ellipsoids

$$z_0 = \{a = a_0, b = b_0, G_1 = l_1 = 0, G_2 = G_{20}, l_2 = l_{20}, G_3 = G_{30}, l_3 = l_{30}\}, \tag{4.1}$$

in the specification of which six parameters are related [3] by four equilibrium equations.

In the parameter space of the family $P^2_{(1)}$ of ellipsoids (4.1) we choose a_0 and b_0 as the coordinates [3].

Note that the equilibrium equations here are the equations

$$\left.\frac{\partial h(\bar{z};(z_0))}{\partial G_2}\right|_{\bar{z}=\bar{z}_0}=\left.\frac{\partial h(\bar{z};(z_0))}{\partial l_2}\right|_{\bar{z}=\bar{z}_0}=\left.\frac{\partial h(\bar{z};(z_0))}{\partial a}\right|_{\bar{z}=\bar{z}_0}=\left.\frac{\partial h(\bar{z};(z_0))}{\partial b}\right|_{\bar{z}=\bar{z}_0}=0 \tag{4.2}$$

$$h(\bar{z};(z_0))=H\Big|_{G_3=G_{30}\left(1+\frac{G_{20}^2-G_2^2}{G_{30}^2}\right)^{1/2},\quad l_3=l_{30}\left(1+\frac{l_{20}^2-l_2^2}{l_{30}^2}\right)^{1/2}}$$

This immediately follows from the conclusions reached in Section 2 and the form of the function H (2.4). The first two of Eqs (4.2) give

$$\frac{a_0^2+c_0^2}{(a_0^2-c_0^2)^2}-\frac{a_0^2+b_0^2}{(a_0^2-b_0^2)^2}=\frac{2a_0b_0}{(a_0^2-b_0^2)^2}\frac{l_{30}}{G_{30}}-\frac{2a_0c_0}{(a_0^2-c_0^2)^2}\frac{l_{20}}{G_{20}}=$$
$$=\frac{2a_0b_0}{(a_0^2-b_0^2)^2}\frac{G_{30}}{l_{30}}-\frac{2a_0c_0}{(a_0^2-c_0^2)^2}\frac{G_{20}}{l_{20}} \tag{4.3}$$

that enables us [3] to determine the relations $l_{30}/G_{30}(a_0, b_0)$ and $l_{20}/G_{20}(a_0, b_0)$ explicitly from the corresponding quadratic equation; in Eqs (4.3) and everywhere henceforth $c_0 = 1/(a_0b_0)$. The last two equations of (4.2), taking (4.3) into account, represent a system of two inhomogeneous equations, linear in G_{20} and G_{30}, all of whose coefficients are explicit functions of a_0 and b_0.

Hence, the equilibrium equations which related the parameters occurring in the specification of ellipsoids (4.1) are such that the functions

$$\frac{l_{30}}{G_{30}}(a_0,b_0),\quad \frac{l_{20}}{G_{20}}(a_0,b_0),\quad G_{20}^2(a_0,b_0), G_{30}^2(a_0,b_0) \tag{4.4}$$

are explicit analytical functions of known form. All that was said in Section 3 regarding the stability of the equilibrium ellipsoids also obviously applies in general form to the family of ellipsoids (4.1). Here, because of the explicit form of the functions (4.4), the analytical specification of the regions S and U in the parameter space $P_{(1)}^2\{a_0, b_0\}$ presents no difficulties from the computational point of view.

The region S is specified by the following two conditions

$$m_{11}^{11}>0,\quad m_{22}^{11}m_{11}^{11}-m_{12}^{11^2}>0$$

where

$$m_{jj}^{11}=\frac{b_0^2+c_0^2}{(b_0^2-c_0^2)^2}-\frac{a_0^2+b_0^2}{(a_0^2-b_0^2)^2}-\frac{2^{2-j}a_0b_0}{(a_0^2-b_0^2)^2}\left(\frac{l_{30}}{G_{30}}\right)^{3-2j},\quad j=1,2$$
$$m_{12}^{11}=\frac{2b_0c_0}{(b_0^2-c_0^2)^2} \tag{4.5}$$

and

$$d^2\left\{\frac{a^2+c^2}{(a^2-c^2)^2}G_2^2+\frac{a^2+b^2}{(a^2-b^2)^2}(G_{20}^2+G_{30}^2-G_2^2)+\frac{a^2+c^2}{(a^2-c^2)^2}l_2^2+\right.$$
$$+\frac{a^2+b^2}{(a^2-b^2)^2}(l_{20}^2+l_{30}^2-l_2^2)+\frac{4ac}{(a^2-c^2)^2}G_2l_2+\frac{4ab}{(a^2-b^2)^2}G_{30}l_{30}\times$$
$$\left.\left.\times\left(1+\frac{G_{20}^2-G_2^2}{G_{30}^2}\right)^{1/2}\left(1+\frac{l_{20}^2-l_2^2}{l_{30}^2}\right)^{1/2}\right\}\right|_{(a,b,G_2,l_2)=(a_0,b_0,G_{20},l_{20})}>0 \tag{4.6}$$

It can be seen that all the coefficients of the quadratic form on the left-hand side of condition (4.6) are functions of a_0, b_0, l_{30}/G_{30}, l_{20}/G_{20}, G_{30}^2 and G_{30}. Substituting the explicit form of relations (4.4) into (4.5) and (4.6), we obtain, in explicit analytical form, the conditions which specify the region S of clearly stable ellipsoids (4.1) in $P_{(1)}^2\{a_0, b_0\}$.

We will now determine the region D from Section 3 in $R_{(1)}^2\{a_0, b_0\}$.

Each of systems (3.1), linearized in the neighbourhood of "its own" equilibrium z_0

$$\delta\dot{\bar{z}}_i = 2\omega(z_0)_{ij}(h^{(2)}(\delta\bar{z};(z_0)))_{jk}\delta\bar{z}_k$$

where $||\omega(z_0)||_{ij}$ is the matrix of bracket (3.2) at the point z_0, can be reduced to the form (3.4) by the simple diagonal replacement

$$\left.\begin{aligned}\{\delta\bar{z}\} \to \{y_{(z_0)}\} = \{y_1 = a,\ y_2 = p_a,\ y_3 = b, y_4 = p_b;\\ y_5 = \frac{\delta G_1}{\sqrt{|G_{30}|}},\ y_6 = \frac{\delta G_2}{\sqrt{|G_{30}|}},\ y_7 = \frac{\delta l_1}{\sqrt{|l_{30}|}},\ y_8 = \frac{\delta l_2}{\sqrt{|l_{30}|}}\end{aligned}\right\}$$

Taking into account the expression obtained for the form $K^{(2)}(y;\ (z_0))$ we conclude that the region D combines four regions, $D^{\alpha\beta}(\alpha, \beta = 1, 2)$, specified from the condition for the restriction of this form to the Lagrange plane in the symplectic manifold $\{y\}$, corresponding to the planes $L^{\alpha\beta}(p_a, p_b, \delta G_\alpha, \delta l_\beta)$ in the initial coordinates to be positive-definite. Using the explicit form of the function $h(\bar{z};\ (z_0))$, defined by the second formula of (4.2), we obtain that these conditions are equivalent to the conditions for the 2×2 matrix $M^{\alpha\beta}$ of the following form to be positive-definite

$$M^{11} = \| m_{ij}^{11} \|, \quad M^{12} = \| m_{ij}^{12} \|, \quad M^{21} = \| m_{ij}^{21} \|, \quad M^{22} = \| m_{ij}^{22} \|$$

where $m_{ij}^{11}(a_0, b_0)$ are functions from condition (4.5),

$$\begin{aligned}
&m_{11}^{12} = m_{11}^{11},\quad m_{12}^{12} = m_{21}^{12} = 0\\
&m_{22}^{12} = \frac{a_0^2 + c_0^2}{(a_0^2 - c_0^2)^2} - \frac{a_0^2 + b_0^2}{(a_0^2 - b_0^2)^2} + \frac{4a_0b_0}{(a_0^2 - b_0^2)^2}\frac{G_{30}}{l_{30}}\left(\frac{l_{20}^2}{l_{30}^2} - \frac{1}{2}\right)\\
&m_{11}^{21} = m_{22}^{12}\,|_{l_{30}\leftrightarrow G_{30},\, l_{20}\leftrightarrow G_{20}},\quad m_{12}^{21} = m_{21}^{21} = 0,\quad m_{22}^{21} = m_{22}^{11}\\
&m_{11}^{22} = m_{11}^{21},\quad m_{22}^{22} = m_{22}^{12}\\
&m_{12}^{22} = m_{21}^{22} = \frac{2a_0c_0}{(a_0^2 - c_0^2)^2} + \frac{2a_0b_0}{(a_0^2 - b_0^2)^2}\frac{G_{20}l_{20}}{G_{30}l_{30}}
\end{aligned} \tag{4.7}$$

It can be seen that $D^{\alpha\beta} \neq \emptyset$

Substituting the explicit relations (4.4) into (4.7) for the elements $m_{ij}^{\alpha\beta}$, we obtain, in explicit analytical form, the conditions which specify the regions $D^{\alpha\beta}$ in the parameter space $P_{(1)}^2\{a_0, b_0\}$

$$m_{11}^{\alpha\beta}(a_0, b_0) > 0,\quad m_{11}^{\alpha\beta}(a_0, b_0)m_{22}^{\alpha\beta}(a_0, b_0) - (m_{12}^{\alpha\beta}(a_0, b_0))^2 > 0 \tag{4.8}$$

Almost all ellipsoids (4.1) $z_0 \in U$ are unstable, where

$$U = \bigcup_{\alpha,\beta=1,2} D^{\alpha\beta} \setminus \bar{S},$$

where the regions $D^{\alpha\beta}$ and S are specified by conditions (4.8) and (4.5)–(4.6) respectively.

Hence, taking into account the observations from Section 3 and because of the appropriate choice of coordinates in the parameter space $P_{(1)}^2$, it turns out to be possible to specify the stability and instability regions in $P_{(1)}^2$ in explicit analytical form.

The stability of the ellipsoids of the first family of Riemann equilibrium ellipsoids $P_{(1)}^2$ has not previously been investigated in the literature by analytical methods.

5. THE STABILITY OF THE EQUILIBRIUM ELLIPSOIDS OF THE SECOND FAMILY OF RIEMANN ELLIPSOIDS

We will now consider the stability of the equilibrium ellipsoids which belong to the second family of Riemann ellipsoids indicated in Section 4 [3]

$$z_0 = \{a = a_0, b = b_0, \dot{a} = \dot{b} = 0, p = q = \omega_1 = \omega_2 = 0, r = \Omega, \omega_3 = f\Omega\} \tag{5.1}$$

or in the variables $\{z\}$ (2.1)

$$z_0 = \{a = a_0, b = b_0, p_a = p_b = 0, l_1 = l_2 = G_1 = G_2 = 0;$$
$$G_3 = \Omega(C_{10} + fC_{20})^{1/2}, \ l_3 = -\frac{2M}{5}a_0 b_0 f\Omega\}$$
$$f > 1, a_0 > c_0 = 1/(a_0 b_0), \quad b_0 > c_0 = 1/(a_0 b_0)$$

The four parameters (a_0, b_0, f, Ω) which occur in the specification of ellipsoids (5.1), are related by the following two equations [3, 4]

$$(1 + d_0^2)\Omega^2 = \frac{3M}{2}\int \frac{\lambda d\lambda}{\tilde{\varphi}_0(\lambda)\sqrt{\varphi_0(\lambda)}}$$

$$a_0 b_0 d_0 \Omega^2 = a_0^2 b_0^2 \frac{3M}{4}\int \frac{d\lambda}{\tilde{\varphi}_0(\lambda)\sqrt{\varphi_0(\lambda)}} - c_0^2 \frac{3M}{4}\int \frac{d\lambda}{(c_0^2 + \lambda)\sqrt{\varphi_0(\lambda)}} \tag{5.2}$$

$$d_0 = \frac{2a_0 b_0}{a_0^2 + b_0^2}(f - 1), \quad \tilde{\varphi}_0(\lambda) = (a_0^2 + \lambda)(b_0^2 + \lambda)$$

$$\varphi_0(\lambda) = (a_0^2 + \lambda)(b_0^2 + \lambda)\left(\lambda + \frac{1}{a_0^2 b_0^2}\right)$$

The set $P_{(2)}^2$ of steady rotations (5.1), like the set $P_{(1)}^2$, is two-dimensional. We choose a_0 and b_0 to be coordinates on this set [3, 4].

The solutions (5.1) of system (1.1)–(1.4) describe steady rotations, for which the motion of the liquid particles is the superposition of a rigid rotation and internal uniformly vortex motions. The vectors of the angular velocity and of the vorticity of the internal motions are directed along the minor axis (c_0) in ellipsoids (5.1).

The Jacobi and Dedekind ellipsoids are limiting special cases of the Riemann ellipsoids (5.1) [3]. The curve specified by the condition $f = 1$ corresponds to a series of Jacobi ellipsoids on the two-dimensional manifold of ellipsoids (5.1), while the ellipsoids (5.1) with $f = \infty$ correspond to a series of Dedekind ellipsoids.

We will now consider the Lyapunov stability of the Riemann ellipsoids (5.1) as particular solutions of system (1.1)–(1.4).

Suppose z_0 is ellipsoids (5.1) with semiaxes a_0 and b_0. Consider the restriction $\bar{h}(\bar{z}; (z_0))$ of the function H (2.4) to that combined level of momentum integral (1.8) and circulation integral (1.9), to which this ellipsoid belongs. Like the function $\bar{h}(\bar{z}; (z_0))$ from Section 4, the function $\bar{h}(\bar{z}; (z_0))$, defined by the second formula of (4.2), depends parametrically on a_0 and b_0, but this dependence is quite different here since the constants of integrals (1.8) and (1.9) for the case of ellipsoids $z_0 \in P_{(2)}^2$ (5.1) are expressed differently in terms of a_0 and b_0 than for the case of the ellipsoids $z_0 \in P_{(1)}^2$ (4.1). Hence, the sufficient condition for the stability of the ellipsoid z_0 (5.1)

$$\bar{h}^{(2)}(\delta\bar{z}; (z_0)) = d^2(\bar{h}(\bar{z}; (z_0)))|_{\bar{z}=\bar{z}_0} > 0 \tag{5.3}$$

(see Section 3), represented in explicit form in the variables a_0 and b_0, is obviously not the same as the corresponding representation of condition (3.3).

It turns out to be very much simpler than (3.3) and reduces, as can easily be shown, to the condition

$$d^2(F(a, b; (z_0)))|_{(a,b)=(a_0 b_0)} > 0 \tag{5.4}$$

where

$$F(a,b;(z_0)) = \bar{h}(\tilde{z};(z_0))|_{p_a=p_b=0,l_1=l_2=G_1=G_2=0} =$$
$$= W + \frac{1}{2} f^2\Omega^2\left(C_2 \frac{a_0^2 b_0^2}{a^2 b^2}\right) + \frac{\Omega^2}{2C_1}\left[C_{10} + f\left(C_{20} - C_2 \frac{a_0 b_0}{ab}\right)\right]^2 \tag{5.5}$$

This conclusion follows from the fact that the form $\bar{h}^{(2)}(\delta\tilde{z};(z_0))$ is the sum

$$\bar{h}^{(2)}(\tilde{z};(z_0))d^2(\bar{h}(\tilde{z};(z_0))|_{a=a_0 b=b_0})|_{(p_a,p_b,l_1,l_2,G_1,G_2)=(0,0,0,0,0,0)} +$$
$$+ d^2(F(a,b;(z_0))|_{(a,b)=(a_0,b_0)} \tag{5.6}$$

where the first form on the right-hand side of (5.6) is positive-definite for all $(a_0, b_0) \in P^2_{(2)}$, which can be shown directly. We will now represent the stability condition (5.4) for the ellipsoids z_0 (5.1) in explicit analytical form in the variables a_0 and b_0.

We reduce the matrix of the quadratic part of the function F (5.5) to the form $\|s_{ij}\|$, where

$$s_{11} = W_{aa} + \frac{3M}{5}\Omega^2(1+d_0^2), \quad s_{22} = W_{bb} + \frac{3M}{5}\Omega^2(1+d_0^2)$$
$$s_{12} = s_{21} = W_{ab} + \frac{6M}{5}\Omega^2 d_0 \tag{5.7}$$

Here

$$W_{aa} = \frac{\partial^2}{\partial a^2}\left(W\left(a, b, \frac{1}{ab}\right)\right)\Bigg|_{(a,b)=(a_0,b_0)}$$

etc. The quantity d_0 is defined by the penultimate formula of (5.2).

Further, it is pertinent to make a certain digression here, which touches on the stability of the Dedekind ellipsoids

$$\left\{a = a_0, b = b_0, p = q = \omega_1 = \omega_2 = 0, r = 0, \omega_3 = \frac{\Omega}{2}\left(\frac{a_0}{b_0} + \frac{b_0}{a_0}\right)\right\} \tag{5.8}$$

each of which is specified [3] by the parameters (a_0, b_0, Ω) of some Jacobi ellipsoid ("conjugate" [3, 4] with respect to the Dedekind ellipsoid considered). Note that the matrix with elements (5.7) for Dedekind ellipsoid (5.8), in which we must now neglect all terms with Ω, $d_0\Omega^2$ and replace $d_0^2\Omega^2$ by Ω^2, change to the same matrix into which the matrix with elements (5.7) for the Jacobi ellipsoid conjugate to (5.8) changes, where the latter is identical with the corresponding matrix obtained previously [5]. The fact that each such matrix is positive-definite [5] therefore proves not only the stability of the corresponding Jacobi ellipsoid but also the stability of the Dedekind ellipsoid "conjugate" to it. Hence, the Dedekind ellipsoids are always stable when defined.

The expressions for the elements s_{ij} (5.7) enable us to determine, in explicit analytical form, in the parameter space $P^2_{(2)}(a_0, b_0)$, specifying the ellipsoids (5.1), the region of those of them to which the certainly stable ellipsoids (5.1) correspond. In fact, bearing in mind the equilibrium equations (5.2), we obtain that the parameters d_0 and Ω can be eliminated from expressions (5.7) for the elements s_{ij} everywhere where they occur, so that these elements turn out to be functions solely of a_0 and b_0 and take the form

$$s_{11} = W_{aa} + 3\frac{2\cdot 3}{5\cdot 4}M^2\int\frac{\lambda d\lambda}{\bar{\varphi}_0(\lambda)\sqrt{\varphi_0(\lambda)}}.$$
$$s_{22} = W_{bb} + 3\frac{2\cdot 3}{5\cdot 4}M^2\int\frac{\lambda d\lambda}{\bar{\varphi}_0(\lambda)\sqrt{\varphi_0(\lambda)}} \tag{5.9}$$

$$s_{12} = s_{21} = W_{ab} + 3\frac{2\cdot 3}{5\cdot 4}M^2\int\left[\frac{a_0 b_0}{\bar{\varphi}_0(\lambda)\sqrt{\varphi_0(\lambda)}} - \frac{1}{[a_0 b_0 + \lambda(a_0^3 b_0^3)]\sqrt{\varphi_0(\lambda)}}\right]d\lambda$$

It can be shown by elementary calculations that $\mathrm{tr}\|s_{ij}\| > 0$ for all a_0 and b_0 (the integrand for s_{11} and s_{22} is a fraction with denominator $a_0^2 b_0^2 \varphi^{5/2}$ and numerator a polynomial in λ, all the coefficients of which are positive). Hence, the necessary and sufficient condition for the matrix with elements (5.9) to be positive-definite is the condition

$$\Delta(a_0, b_0) = s_{11}(a_0, b_0)s_{22}(a_0, b_0) - s_{12}^2(a_0, b_0) > 0 \tag{5.10}$$

In other words, inequality (5.10) is the analytical representation of condition (5.4) and therefore defines a set of stable ellipsoids (5.1).

We will now consider the problem of determining the instability region in parameter space $P_{(2)}^2$ of ellipsoids (5.2). The condition for the first quadratic form in the variables $(p_a, p_b, l_1, l_2, G_1, G_2)$, written on the right-hand side of relation (5.6), to be positive-definite denotes that all the ellipsoids (5.1) automatically fall in the region D from Section 3. Hence, taking into account Section 3 and the fact that the stability region S is specified in the space $P_{(2)}^2\{a_0, b_0\}$ by condition (5.10), it immediately follows that a condition opposite to condition (5.10) specifies the region U in the space $P_{(2)}^2\{a_0, b_0\}$ such that almost all the ellipsoids (5.1) belonging to it are necessarily unstable.

In fact, all the equilibrium ellipsoids (5.1) from the region specified by a condition opposite to condition (5.10) are unstable, and this conclusions can be obtained in this case without using the remark from Section 3.

In fact, let us assume that any ellipsoid (5.2) with semiaxes a_0 and b_0 is stable. Then, it is also stable like the solution of the system obtained from system (1.1)–(1.4), if we put $l_1 = l_2 = G_1 = G_2 = 0$ (in the initial variables $p = q = \omega_1 = \omega_2 = 0$), $\Sigma G_i^2 = \Sigma G_{i0}^2$, $\Sigma l_i^2 = \Sigma l_{i0}^2$, i.e. a system with phase space of dimension 4 in $\{a, b, p_a, p_b\}$. This latter system is defined since, as directly follows from the form of Eqs (1.1)–(1.4), the manifold $\{p = q = \omega_1 = \omega_2 = 0\}$ and, of course, also the manifold

$$N = \{z : l_1 = l_2 = G_1 = G_2 = 0;\ l_3 = l_{30};\ G_3 = G_{30}\},$$

is invariant for system (1.1)–(1.4). This system, with phase space N, is Hamiltonian (Section 2) for all a_0 and b_0, and moreover, is natural, since the Hamiltonian in it is the sum of the "potential" energy and the "kinetic" energy – a positive-definite function of the momenta p_a and p_b.

We can therefore use the converse of Lagrange's theorem here. It remains to note that the matrix of the quadratic part of the function, which is the potential energy in a system on the manifold N at the point $(a_0, b_0, 0,0)$, is identical with the matrix with elements (5.9). In other words, any non-degenerate $(\Delta(a_0, b_0) \neq 0)$ ellipsoid (5.1), the parameters a_0 and b_0 of which do not satisfy condition (5.10), is unstable even in the class of perturbations for which the orientation of the angular velocity and vorticity vectors remain unchanged and the values of the momentum integral (1.8) and circulation integral (1.9) remain the same as for unperturbed motion, i.e. for the equilibrium ellipsoid considered, and all the more it is Lyapunov stable as the solution of the initial system (1.1)–(1.4).

Hence, the answer to the question of the stability with respect to almost each ellipsoid (5.1), with the exception, possibly, of the bifurcation ellipsoids, can be obtained analytically.

It should noted that problems related to ellipsoidal figures of equilibrium were of interest not only to the nineteenth century classics. For example, the so-called virial method of investigating hydrodynamic equations [3], which is essentially the method of moments, was proposed, where stability was understood to mean stability in the first approximation of the given steady rotation as solutions of the first-, second-, third-, etc. order virial equations. It has been shown [3] that the second-order virial equations are equivalent to system (1.1)–(1.4). Hence, for example, the Riemann ellipsoids (5.1), which are unstable in the first approximation (i.e. those for which $\Delta(a_0, b_0)<0$, where Δ is a function from condition (5.10)), are also unstable in the terminology used by Chandrasekhar [3], who called them "unstable with respect to the second harmonics", in this case, the instability region is defined in [3] by direct linearization of the equations of system (1.1)–(1.4), which requires extremely complicated calculations.

If we are talking about "stable" equilibrium ellipsoids, stability in the first approximation in this case (all the roots are pure imaginary and there are no less than two zero roots) naturally does not guarantee the Lyapunov stability which, therefore, needs to be proved here. This can only be done using Lyapunov's second method.

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